

# Representing and Computing the B-Derivative of the Piecewise-Differentiable Flow of a Class of Nonsmooth Vector Fields

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*This paper concerns first-order approximation of the piecewise-differentiable flow generated by a class of nonsmooth vector fields. Specifically, we represent and compute the Bouligand (or B-)derivative of the piecewise-differentiable flow generated by a vector field with event-selected discontinuities. Our results are remarkably efficient: although there are factorially many “pieces” of the derivative, we provide an algorithm that evaluates its action on a tangent vector using polynomial time and space, and verify the algorithm’s correctness by deriving a representation for the B-derivative that requires “only” exponential time and space to construct. We apply our methods in two classes of illustrative examples: piecewise-constant vector fields and mechanical systems subject to unilateral constraints. [DOI: 10.1115/1.4054481]*

## 1 Introduction

First-order approximations, i.e., derivatives—are a foundational tool for analysis and synthesis in smooth dynamical and control systems. For instance, derivatives play a crucial role in: stability via spectral [1, Ch. 8.3] or Lyapunov [2, Ch. 5] methods; controllability via linearization [1, Ch. 8.7] or Frobenius/Chow [2, Ch. 8/Ch. 11] techniques; optimality via stationarity [3, Ch. 1] or Pontryagin [4, Ch. 1] principles; identifiability via adaptation [5, Ch. 2] or Expectation-Maximization [6, Ch. 10] methods. These tools all depend on the existence of a computationally amenable representation for the first-order approximation of smooth system dynamics, namely, the Fréchet (or F-)derivative of the system’s smooth flow [7, Ch. 5.6], which is a continuous linear function of tangent vectors.<sup>2</sup>

By definition, nonsmooth systems do not generally enjoy existence (let alone computational amenability) of first-order approximations. Restricting to the class of (so-called Ref. [8, Def. 1, 2]) event-selected  $C^r$  ( $EC^r$ ) vector fields that (i) are smooth except along a finite number of surfaces of discontinuity and (ii) preclude sliding [9,10] or branching [11, Def. 3.11] through a transversality condition, we obtain flows that are piecewise-differentiable [8, Thm. 4] (specifically, piecewise- $C^r$  ( $PC^r$ ) [12, Ch. 4.1]). By virtue of their piecewise differentiability, these flows admit a first-order approximation, termed the Bouligand (or B-)derivative, which derivative is a continuous piecewise-linear function of tangent vectors [12, Ch. 3, 4]. This paper is concerned with the efficient representation and computation of this piecewise-linear first-order approximation.

Our contributions are twofold: (i) we construct a representation for the B-derivative of the  $PC^r$  flow generated by an  $EC^r$  vector field; (ii) we derive an algorithm that evaluates the B-derivative on a given tangent vector. Although there are factorially many “pieces” of the derivative, we (i) represent it using exponential time and space and (ii) compute it using polynomial time and space. In an effort to make our results as accessible and useful as

possible, we provide a concise summary of the algorithm in Sec. 2 and apply our methods in Sec. 3 before rehearsing the technical background in Sec. 4 needed to derive the representation in Sec. 5 and verify the algorithm’s correctness in Sec. 6.

We emphasize that our methods are most useful when there are more than two surfaces of discontinuity, as representation and computation of first-order approximations in the one- and two-surface cases have been investigated extensively [13–18], and these cases do not benefit from the complexity savings touted above. Previously, we established existence of the piecewise-linear first-order approximation of the flow [8, Rem. 1] and provided an inefficient scheme to evaluate each of its “pieces” [8, Sec. 7] in the presence of an arbitrary number of surfaces of discontinuity. To the best of our knowledge, this paper contains the first representation for the B-derivative of the  $PC^r$  flow of a general  $EC^r$  vector field and polynomial-time algorithm to compute it.

## 2 Algorithm

The goal of this paper is to obtain an algorithm that efficiently computes the derivative of a class of nonsmooth flows. This computational task and our solution are easy to describe, yet verifying the algorithm’s correctness requires significant technical overhead. Thus, the remainder of this section will be devoted to specifying the algorithm and the problem it solves using minimal notation and terminology. Sections 4–6 will provide technical details—which may be of interest in their own right—that prove the algorithm is correct.

**2.1 Problem Statement.** Given a vector field  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  and a trajectory  $x : [0, \infty) \rightarrow \mathbb{R}^d$  satisfying<sup>3</sup>

$$\forall t \geq 0: x_t = x_0 + \int_0^t F(x_\tau) d\tau \quad (1)$$

our goal is to approximate how  $x_t$  varies with respect to  $x_0$  to first order for a given  $t > 0$ . Formally, with  $\phi : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  denoting the flow of  $F$  satisfying

<sup>3</sup>In this section (only), we will denote time dependence using subscripts rather than parentheses to minimize the notational overhead.

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<sup>2</sup>We emphasize both properties of the classical derivative since the generalized derivative we consider retains one while relaxing the other.

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$$\forall t \geq 0, x_0 \in \mathbb{R}^d: \phi_t(x_0) = x_0 + \int_0^t F(\phi_\tau(x_0)) d\tau \quad (2)$$

our goal is to evaluate the directional derivative  $D\phi_t(x_0; \delta x_0)$  given  $t > 0, \delta x_0 \in T_{x_0}\mathbb{R}^d$

$$D\phi_t(x_0; \delta x_0) = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} (\phi_t(x_0 + \alpha \delta x_0) - \phi_t(x_0)) \quad (3)$$

Specifically, we seek to evaluate this derivative for vector fields that are smooth everywhere except a finite collection of surfaces where they are allowed to be discontinuous. We will first recall how to obtain the derivative in the presence of zero (Sec. 2.2) and one (Sec. 2.3) surfaces of discontinuity before presenting our algorithm, which is applicable in the presence of an arbitrary number of surfaces of discontinuity (Sec. 2.4).

**2.2 Continuously Differentiable Vector Field.** If  $F$  is continuously differentiable on the trajectory  $x$ , the derivative  $\delta x_t = D\phi_t(x_0; \delta x_0)$  satisfies the linear time-varying *variational equation* [19, Appendix B]

$$\forall t \geq 0: \delta x_t = \delta x_0 + \int_0^t DF(x_\tau) \cdot \delta x_\tau d\tau \quad (4)$$

where  $\delta x_t = D\phi_t(x_0; \delta x_0)$  can be approximated to any desired precision in polynomial time by applying numerical simulation algorithms [19, Ch. 4] to Eqs. (1) and (4).

**2.3 Single Surface of Discontinuity.** If  $F$  is continuously differentiable everywhere except a smooth codimension-1 submanifold  $H \subset \mathbb{R}^d$  that intersects the trajectory  $x$  transversally at only one point  $x_s, s \in (0, t)$ , the continuous-time equation Eq. (4) is augmented by the discrete-time update [13, Eq. (58)]

$$\delta x_s^+ = \left( I_d + \frac{(F^+ - F^-) \cdot \eta^\top}{\eta^\top \cdot F^-} \right) \cdot \delta x_s^- = M \cdot \delta x_s^- \quad (5)$$

where  $\delta x_s^\pm = \lim_{\tau \rightarrow s^\pm} \delta x_\tau$  and  $F^\pm = \lim_{\tau \rightarrow s^\pm} F(x_\tau)$  denote the limiting values of  $\delta x_\tau$  and  $F(x_\tau)$  at  $s$  from the right (+) and left (-) and  $\eta \in \mathbb{R}^d$  is any vector orthogonal to surface  $H$  at  $x_s$ ;  $M \in \mathbb{R}^{d \times d}$

is termed the *saltation matrix* [17, Eq. (2.76)], [20, Eq. (7.65)]. The desired derivative is

$$D\phi_t(x_0; \delta x_0) = D\phi_{t-s}(x_s) \cdot M \cdot D\phi_s(x_0) \cdot \delta x_0 \quad (6)$$

where  $D\phi_{t-s}(x_s), D\phi_s(x_0) \in \mathbb{R}^{d \times d}$  can be approximated by simulating Eq. (2), Eq. (4) since the flow is smooth away from time  $s$ . Computing the saltation matrix  $M$  requires  $O(d^2)$  time and space, but evaluating its action on  $\delta x_s^-$  in Eq. (5) requires only  $O(d)$  time and space.

**2.4 Multiple Surfaces of Discontinuity.** If  $F$  is continuously differentiable everywhere except a finite set of smooth codimension-1 submanifolds  $\{H_j\}_{j=1}^n$  that intersect the trajectory  $x$  transversally at only one point  $x_s$  (see Fig. 1(a) for an illustration when  $n=2$ ),  $s \in (0, t)$ , we showed in Ref. [8, Eq. (65)] that the discrete-time update Eq. (5) is applied once for each surface. However, the order in which the updates are applied, and the limiting values of the vector field used to determine each update's saltation matrix, depend on  $\delta x_0$ . If the surfaces intersect transversally, there are  $n!$  different saltation matrices determined by  $2^n$  vector field values, so considering all update orders requires factorial time and space. To make these observations precise and specify the notation employed in Figs. 1 and 2, we formally define the class of nonsmooth vector fields considered in this paper [8, Defs. 1, 2].

**DEFINITION 1. (Event-selected  $C^r$  ( $EC^r$ ) vector field)** A vector field  $F: D \rightarrow TD$  defined on an open domain  $D \subset \mathbb{R}^d$  is event-selected  $C^r$  with respect to  $h \in C^r(U, \mathbb{R}^n)$  at  $\rho \in \mathbb{R}^d$  if  $U \subset D$  is an open neighborhood of  $\rho$  and:

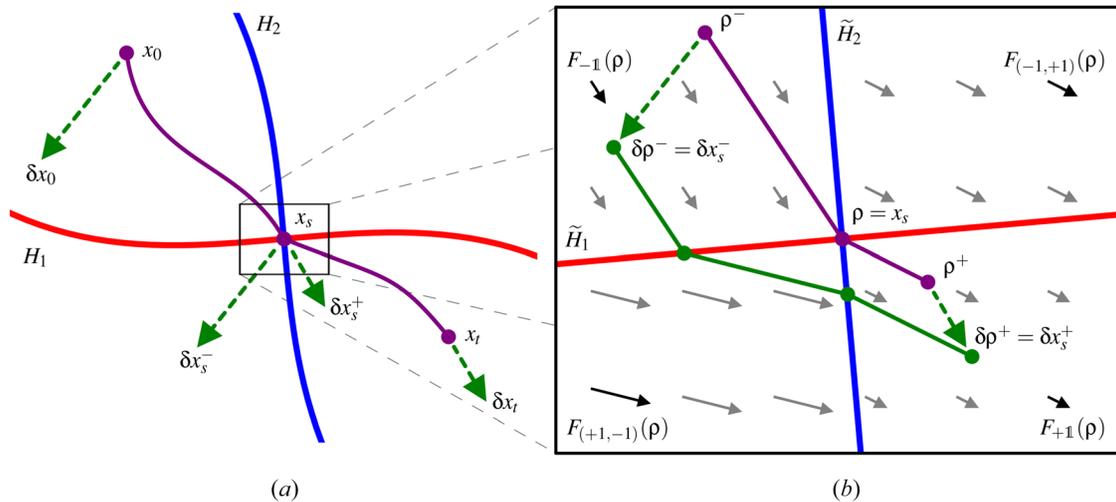
1. (event functions) there exists  $f > 0$  such that  $Dh(x) \cdot F(x) \geq f$  for all  $x \in U$ ;
2. (smooth extension) for all  $b \in \{-1, +1\}^n = B_n$ , with

$$D_b = \{x \in U : b_j(h_j(x) - h_j(\rho)) \geq 0\} \quad (7)$$

$F|_{\text{Int}D_b}$  admits a  $C^r$  extension  $F_b: U \rightarrow TU$ .

Our algorithms in Fig. 2 compute

$$\delta x_s^+ = \delta \rho^+ = B(\delta \rho^-) = B(\delta x_s^-) \quad (8)$$



**Fig. 1 Variational dynamics that determine B-derivative of planar  $EC^r$  vector field's  $PC^r$  flow (8).** (a) Vector field  $F: \mathbb{R}^2 \rightarrow T\mathbb{R}^2$  is smooth everywhere except smooth codimension-1 submanifolds  $H_1, H_2 \subset \mathbb{R}^2$  that intersect transversally at  $x_s \in \mathbb{R}^2$ , generating a piecewise-differentiable flow  $\phi: [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfying  $\phi_\tau(x_0) = x_\tau$  for all  $\tau \in [0, t]$ , i.e.,  $F$  is  $EC^r$  and  $\phi$  is  $PC^r$  [8]. The B-derivative  $D\phi_t(x_0; \delta x_0) = \delta x_t$  is determined as in Eq. (10) by the continuous-time variational dynamics  $\delta \dot{x}_\tau = DF(x_\tau) \cdot \delta x_\tau$  and the discrete-time variational dynamics  $\delta x_s^+ = B(\delta x_s^-)$ . The algorithms in Fig. 2 evaluate the piecewise-linear function  $B$  using the auxiliary system in (b) determined by the tangent planes  $\tilde{H}_1, \tilde{H}_2$  and vector field limits  $F_b(\rho)$  in Eq. (9) for  $b \in \{(-1, -1), (+1, -1), (-1, +1), (+1, +1)\} = \{-1, +1\}^2$ .

**Algorithm 1**  $\delta\rho^+ \leftarrow B(\delta\rho^-, \eta, \Gamma)$ 

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1:  $\delta t \leftarrow 0 \in \mathbb{R}$ 
2:  $\delta\rho^+ \leftarrow \delta\rho^- \in \mathbb{R}^d$ 
3:  $b \leftarrow -\mathbb{1} \in \{-1, +1\}^n$ 
4: while  $b \neq +\mathbb{1}$  do
5:   for  $j \in \{1, \dots, n\}$  do
6:      $\tau_j \leftarrow -(\eta_j^\top \cdot \delta\rho^+) / (\eta_j^\top \cdot \Gamma(b))$ 
7:    $j^* \leftarrow \arg \min_{j \in \{1, \dots, n\}} \{\tau_j : b_j < 0\}$ 
8:    $\delta t \leftarrow \delta t + \tau_{j^*}$ 
9:    $\delta\rho^+ \leftarrow \delta\rho^+ + \tau_{j^*} \cdot \Gamma(b)$ 
10:   $b_{j^*} \leftarrow +1$ 
11: return  $\delta\rho^+ - \delta t \cdot \Gamma(+\mathbb{1})$ 

```

**Algorithm 2** def B(dx, e, G):

```

1: dt = 0
2: dx = np.array(dx)
3: b = -np.ones(len(e), dtype=np.int)
4: while np.any(b < 0) :
5:   tau = -np.dot(e, dx) / np.dot(e, G(b))
6:   tau[b > 0] = np.inf
7:   j = np.argmin(tau)
8:   dt += tau[j]
9:   dx += tau[j] * G(b)
10:  b[j] = +1
11: return dx - dt * G(b) # b == [+1, ..., +1]

```

**Fig. 2** Algorithms that evaluate the B-derivative of an  $EC^r$  vector field's  $PC^r$  flow written in pseudo-code (left) and Python [21] source code (right; requires import numpy as np [22]). These algorithms apply at a point  $\rho \in \mathbb{R}^d$  where a vector field  $F: \mathbb{R}^d \rightarrow T\mathbb{R}^d$  is event-selected  $C^r$  with respect to  $n$  surfaces (see Fig. 1 for an illustration when  $d = n = 2$ ), and assume the following data is given: *tangent direction, surface normals at  $\rho$ , vector field limits (9),  $\delta\rho^- \in T_\rho\mathbb{R}^d$ ,  $\eta = \{\eta_j\}_{j=1}^n \subset \mathbb{R}^d$ ,  $\Gamma: \{-1, +1\}^n \rightarrow \mathbb{R}^d$ ,  $dx$ —array,  $dx.shape == (d,)$ ;  $e$ —array,  $e.shape == (n, d)$ ;  $G$ —function,  $G(b).shape == (d,)$ .*

given  $\delta\rho^- = \delta x_s^- \in \mathbb{R}^d$ , normals  $\{\eta_j = Dh_j(\rho)\}_{j=1}^n \subset \mathbb{R}^d$  at  $x_s$  to surfaces  $\{H_j = h_j^{-1}(\rho)\}_{j=1}^n$ , and a function  $\Gamma: \{-1, +1\}^n \rightarrow \mathbb{R}^d$  that evaluates limits of  $F$  at  $\rho = x_s$

$$\forall b \in \{-1, +1\}^n: \Gamma(b) = F_b(\rho) \quad (9)$$

using the piecewise-constant dynamics illustrated in Fig. 1(b), which are the discrete-time analog of the continuous-time variational dynamics Eq. (4). Overall, the desired derivative is

$$D\phi_t(x_0; \delta x_0) = D\phi_{t-s}(x_s) \cdot B(D\phi_s(x_0) \cdot \delta x_0) \quad (10)$$

where  $B: T_\rho\mathbb{R}^d \rightarrow T_\rho\mathbb{R}^d$  is the continuous piecewise-linear function defined by our algorithms in Fig. 2. Our algorithms require  $O(n^2d)$  time and  $O(d)$  space to evaluate the directional derivative Eq. (3).

Assuming for the moment that these algorithms are correct, we emphasize that they achieve a dramatic reduction in the computational complexity of evaluating the B-derivative—from factorial to low-order polynomial—relative to naïve enumeration of all pieces of the B-derivative. However, despite the apparent simplicity of our algorithms (computationally and conceptually), verifying their correctness requires significant technical effort; the bulk of this paper is devoted to this verification task.

### 3 Applications

To illustrate and validate our methods, we apply the algorithm from Sec. 2 to piecewise-constant vector fields in Sec. 3.1 and mechanical systems subject to unilateral constraints in Sec. 3.2.

**3.1 Piecewise-Constant Vector Field.** Consider the vector field  $F: \mathbb{R}^d \rightarrow T\mathbb{R}^d$  defined by

$$\dot{x} = F(x) = \mathbb{1} + \Delta(\text{sign}(x)) \quad (11)$$

where  $\Delta: B_d \rightarrow \mathbb{R}^d$ , so long as all components of all vectors specified by  $\Delta$  are larger than  $-1$ , i.e.,  $\min_{b \in B_d} [\Delta(b)]_j > -1$ ,  $F$  is event-selected  $C^\infty$  with respect to the identity function  $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined by  $h(q) = q$ . We regard Eq. (11) as a canonical form for piecewise-constant event-selected  $C^\infty$  vector fields that are discontinuous across  $d$  subspaces, since any such vector field can be obtained by applying a linear change-of-coordinates to Eq. (11). In what follows, we focus on the trajectory that passes through the origin  $\rho = 0$ , which lies at the intersection of  $d$

surfaces of discontinuity for  $F$ . With  $\rho^- = \rho - \frac{1}{2}F_{-1}(\rho)$ ,  $\rho^+ = \rho + \frac{1}{2}F_{+1}(\rho)$ , we note that  $\rho^-$  flows to  $\rho^+$  through  $\rho$  in 1 (one) unit of time.

Our goal is to compute  $D_x\phi(1, \rho^-; \delta\rho^-) \in T_{\rho^+}\mathbb{R}^d$  for a given  $\delta\rho^- \in T_{\rho^-}\mathbb{R}^d$ . In the general case, the desired derivative is piecewise linear with (up to)  $d!$  distinct pieces, providing a general test. In the special case where  $\Delta(b) = -\delta \cdot b$  for all  $b \in B_d$ ,  $|\delta| < 1$ , the desired derivative is linear [8, Eq. (86)]

$$D_x\phi(1, \rho^-; \delta\rho^-) = \frac{1 - \delta}{1 + \delta} \cdot \delta\rho^- \quad (12)$$

providing a closed-form expression for comparison. Figure 3 illustrates results from both cases with  $d = 2$ .

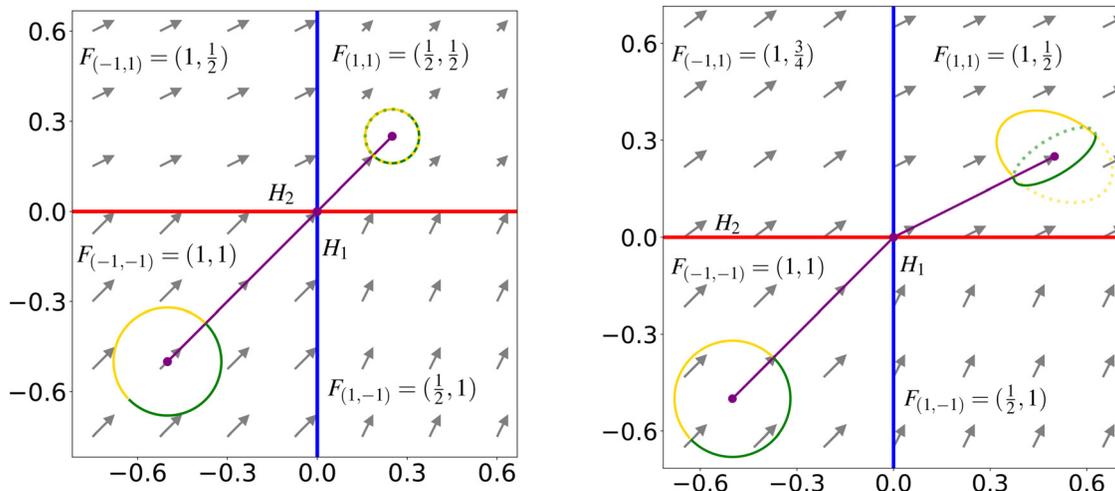
#### 3.2 Mechanical systems Subject to One-Sided Constraints.

Consider a mechanical system whose configuration is subject to *unilateral* (i.e., one-sided) constraints. The dynamics of such systems have been studied extensively using the formalisms of complementarity [23, Sec. 3], measure differential inclusions [24, Sec. 3], hybrid systems [25, Sec. 2.4, 2.5], and geometric mechanics [26, Sec. 3]. Regardless of the chosen formalism, in a coordinate chart  $Q \subset \mathbb{R}^d$ , the dynamics governing  $q$  takes the form

$$M(q)\ddot{q} = f(q, \dot{q}) \text{ subject to } a(q) \geq 0 \quad (13)$$

where:  $M(q) \in \mathbb{R}^{d \times d}$  specifies the kinetic energy metric;  $f(q, \dot{q}) \in \mathbb{R}^d$  specifies the internal, applied, and Coriolis forces;  $a(q) \in \mathbb{R}^n$  specifies the *unilateral constraints* (we interpret  $a(q) \geq 0$  componentwise); and we assume in what follows that  $M$ ,  $f$ , and  $a$  are smooth functions. Different formalisms enforce the constraint  $a(q) \geq 0$  in Eq. (13) differently, so we consider several cases through Secs. 3.2.1–3.2.4.

**3.2.1 Rigid Constraints Yield Discontinuous Flows.** If constraints are enforced *rigidly* as in Refs. [23–25], meaning that they must be satisfied exactly, then the velocity must undergo impact (i.e., change discontinuously) whenever  $\dot{q} \in T_qQ$  is such that  $a_j(q) = 0$  and  $Da_j(q) \cdot \dot{q} < 0$  for some  $j \in \{1, \dots, n\}$  [23, Sec. 2, 25, Eq. (23), 24, Eq. (23)]. Unfortunately for our purposes, these discontinuities in the state vector  $x = (q, \dot{q})$  cannot be modeled using an event-selected  $C^r$  vector field  $\dot{x} = F(x)$ , and the flow of such systems is generally discontinuous (although we note that the flow can be  $PC^r$  at nonimpact times if the constraint surfaces intersect orthogonally [27], i.e., if the surface normals are orthogonal with respect to the inverse of the kinetic energy metric [24, Theorem 20]).



**Fig. 3** B-derivative of planar instance of vector field from Sec. 3.1 in linear (left) and piecewise-linear (right) cases. The vector field  $F$  defined in Eq. (11) is piecewise constant and discontinuous across the coordinate hyperplanes  $H_1, H_2$ , generating a piecewise-differentiable flow  $\phi$  with B-derivative  $B$ . (left) The B-derivative is linear in the special case defined by Eq. (12). (right) The B-derivative is continuous and piecewise linear in general, so a ball of initial conditions flows to a piecewise ellipsoid (solid lines).

**3.2.2 Soft Conservative Constraints Yield Lipschitz-Continuous Vector Fields,  $C^1$  Flows.** We now consider the formalism in Ref. [26] that “softens” (i.e., approximately enforces) rigid constraints  $a_j(q) \geq 0$  by augmenting the potential energy with *penalty functions*  $\{v_j\}_{j=1}^n$  that scale quadratically with the degree of constraint violation [26, Eq. (12)]

$$\forall j \in \{1, \dots, n\} : v_j(q) = \begin{cases} 0, & a_j(q) \geq 0 \\ \frac{1}{2} \kappa_j a_j^2(q), & a_j(q) < 0 \end{cases} \quad (14)$$

In essence, each rigid constraint  $a_j(q) \geq 0$  is replaced by a spring with stiffness  $\kappa_j$ , leading to the unconstrained dynamics [26, Eq. (14)]

$$\begin{aligned} M(q)\ddot{q} &= f(q, \dot{q}, u) - \sum_{j=1}^n Dv_j(q)^\top \\ &= f(q, \dot{q}, u) - \sum_{j=1}^n \{(\kappa_j a_j(q)) \cdot Da_j(q)^\top : a_j(q) < 0\} \end{aligned} \quad (15)$$

As shown by [28, Thm. 3], trajectories of Eq. (15) converge to those of Eq. (13) in the rigid limit (i.e., as stiffnesses go to infinity). Importantly for our purposes, the dynamics in Eq. (15) can be modeled using an event-selected vector field along trajectories that pass transversally through the constraint surfaces, whence our algorithms can compute the B-derivative of the flow. However, the vector field Eq. (15) in this case is (locally Lipschitz) continuous; hence, the B-derivative is trivial (all nonidentity terms in Eq. (57) are zero), where the flow is continuously differentiable ( $C^1$ ).

**3.2.3 Soft Dissipative Constraints Yield  $EC^r$  Vector Fields,  $C^1$  Flows.** We now augment the unconstrained dynamics Eq. (15) with dissipation as in [26]

$$\begin{aligned} M(q)\ddot{q} & \\ &= f(q, \dot{q}, u) - \sum_{j=1}^n \{\alpha_j(q, \dot{q}) \cdot Da_j(q)^\top : a_j(q) < 0\} \end{aligned} \quad (16)$$

where  $\alpha_j(q, \dot{q}) = \kappa_j a_j(q) + \beta_j Da_j(q) \cdot \dot{q}$ ; in essence, each constraint penalty is augmented by a spring damper that is only active when the constraint is violated as in studies involving contact with complex geometry [29] or terrain [30]. The dynamics in Eq. (16) can be modeled using an event-selected vector field along trajectories that pass transversally through the constraint surfaces, and the vector field is discontinuous along the constraint surfaces.

However, we can show that the flow of Eq. (16) is continuously differentiable ( $C^1$ ) along *any* trajectory that passes transversally through constraint surfaces. Indeed, letting  $x = (q, \dot{q})$  denote the state of the system so that  $\dot{x} = (\dot{q}, \ddot{q}) = F(x)$  is determined by Eq. (16), the saltation matrix Eq. (5) associated with each constraint  $a_j$  has the form

$$I + \frac{1}{Da_j(q) \cdot \dot{q}} \begin{bmatrix} 0 \\ \pm \alpha_j(q, \dot{q}) \end{bmatrix} [Da_j(q) \quad 0] \quad (17)$$

where the sign in the column vector is determined by whether the constraint is activating (–) or de-activating (+). Since matrices of the form in Eq. (17) commute, the saltation matrices associated with simultaneous activation and/or de-activation of multiple constraints are all equal, whence the flow of Eq. (16) is continuously differentiable ( $C^1$ ) along any trajectory that passes transversally through constraint surfaces.

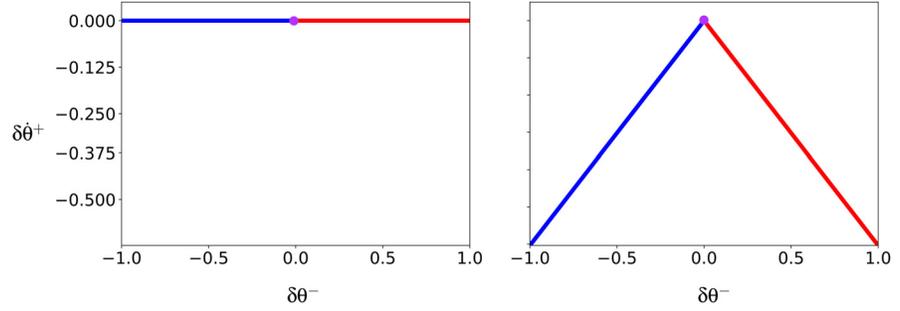
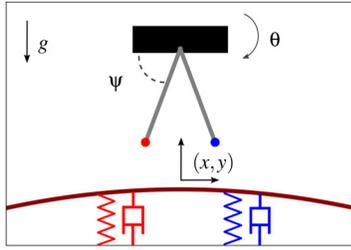
**3.2.4 Example (Vertical-Plane Biped).** To ground the preceding observations, we consider the vertical-plane biped illustrated in Fig. 4(left) that falls under the influence of gravity toward a substrate. The biped body has mass  $m$  and moment-of-inertia  $J$ ; we let  $(x, y) \in \mathbb{R}^2$  denote the position of its center-of-mass in the plane and  $\theta \in S^1$  denote its rotation. Two rigid massless limbs of length  $\ell$  protrude at an angle of  $\pm\psi$  with respect to vertical from the body’s center-of-mass above a smooth substrate whose height is a quadratic function of horizontal position, yielding unilateral constraints

$$\begin{aligned} a_1(x, y, \theta) &= -y - (x + \ell \cos(\theta - \psi))^2 - \ell \sin(\theta - \psi) \\ a_2(x, y, \theta) &= -y - (x + \ell \cos(\theta + \psi))^2 - \ell \sin(\theta + \psi) \end{aligned} \quad (18)$$

We consider the smoothness of the system’s flow along a trajectory that activates both constraints simultaneously. The formalism in Eq. (16) yields continuously differentiable flow for this system as illustrated in Fig. 4 (center).

To obtain a flow that is piecewise differentiable but *not* continuously differentiable, we modify the damping coefficients in Eq. (16) using the following logic<sup>4</sup>:  $\beta_1 = \beta_2 = \frac{1}{2}$  if  $a_1(q) < 0$  or  $a_2(q) \geq 0$  (exclusive or);  $\beta_1 = \beta_2 = 1$  if  $a_1(q) < 0$  and  $a_2(q) < 0$ . The saltation matrices obtained from different sequences of

<sup>4</sup>Although we introduce this logic purely for illustrative purposes, we note that nontrivial dependence of forcing on the set of active constraints could be implemented physically using clutches [31] or actuators [32].



**Fig. 4** Vertical-plane biped, a 3-degree-of-freedom mechanical system subject to unilateral constraints (Sec. 3.2.4), consists of a planar body with two rigid massless legs falling under the influence of gravity toward a substrate (left). The system's flow can be  $C^1$  (center) or  $PC^1$  (right) depending on how forces vary as limbs contact substrate.

constraint activations (left foot reaches substrate before right foot or vice versa) are distinct

$$M_{(\text{left}, \text{right})} - M_{(\text{right}, \text{left})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4\beta \cos(\psi) & 0 & -2\beta(\sin(2\psi) + \cos(\psi)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

The piecewise-linear B-derivative of the system's flow is illustrated in Fig. 4 (right). Source code for this example is provided at the website<sup>5</sup>.

## 4 Background

To verify correctness of the algorithms specified in Sec. 2, we utilize the representation of piecewise-affine functions from Ref. [33], elements of the theory of piecewise-differentiable functions from Ref. [12], and results about the class of nonsmooth flows under consideration from Ref. [8]. In an effort to make this paper self-contained (i.e., to save the reader from needing to cross-reference multiple citations to follow our derivations), we include a substantial amount of background details in this section. The expert reader may wish to skim or skip this section, returning only if questions arise in Sec. 5 or Sec. 6.

**4.1 Polyhedral Theory.** We let  $0_d \in \mathbb{R}^d$  denote the vector of zeros,  $1_n \in \mathbb{R}^n$  the vector of ones, and  $I_d \in \mathbb{R}^{d \times d}$  the identity matrix; when dimensions are clear from context, we suppress subscripts. The vectorized signum function  $\text{sign} : \mathbb{R}^d \rightarrow \{-1, +1\}^d$  is defined for all  $x \in \mathbb{R}^d, j \in \{1, \dots, d\}$  by

$$[\text{sign}(x)]_j = \text{sign}(x_j) = \begin{cases} -1, & x_j < 0 \\ +1, & x_j \geq 0 \end{cases} \quad (20)$$

If  $A \in \mathbb{R}^{\ell \times m}$  and  $B \in \mathbb{R}^{m \times n}$  then  $A \cdot B \in \mathbb{R}^{\ell \times n}$  denotes matrix multiplication. Given a subset  $S \subset \mathbb{R}^d$ , we let  $\text{aff } S$ ,  $\text{cone } S$ ,  $\text{conv } S$  denote the *affine span*, *cone span*, and *convex hull* of  $S$ , respectively [12, Sec. 2.1.1]. The *dimension* of a convex set  $S$  is defined to be the dimension of its affine span,  $\dim S = \dim \text{aff } S$ . A nonempty set  $S \subset \mathbb{R}^d$  is called a *polyhedron* [12, Sec. 2.1.2] if there exists  $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m$  such that  $S = \{x \in \mathbb{R}^d : A \cdot x \leq b\}$ ; note that  $S$  is closed and convex. The linear subspace  $\mathcal{L} = \{x \in \mathbb{R}^d : A \cdot x = 0\}$  is called the *lineality space* of  $S$ .

**4.2 Piecewise-Affine Functions.** We will represent a piecewise-affine function using a *triangulation*  $(Z^-, Z^+, \Delta)$  [33, Sec. 3.1] that consists of a combinatorial simplicial complex

$\Delta$  whose vertex set is in 1-to-1 correspondence with each of the finite sets of vectors  $Z^- \subset \mathbb{R}^d, Z^+ \subset \mathbb{R}^c$ . For our purposes,<sup>6</sup> a *combinatorial simplicial complex*  $\Delta$  is a collection of finite sets  $\Delta = \{\Delta_\omega\}_{\omega \in \Omega}$  such that  $S \subset \Delta_\omega \Rightarrow S \in \Delta$  for all  $\omega \in \Omega$ ; we call  $\cup_{\omega \in \Omega} \Delta_\omega$  the *vertex set* of  $\Delta$ . We assume that, for every  $\omega \in \Omega$ , the collections of vectors  $Z_\omega^\pm \subset Z^\pm$  determined by  $\Delta_\omega$  are *affinely independent* [33, Sec. 2.1.1] so that  $\Delta_\omega^\pm = \text{conv} Z_\omega^\pm$  are  $((\Delta_\omega) - 1)$ -dimensional geometric simplices [33, Claim 2.9] where  $\Delta_\omega^- \subset \mathbb{R}^d, \Delta_\omega^+ \subset \mathbb{R}^c$ . We assume further that, for every  $\omega, \omega' \in \Omega$ , the collections of vectors  $Z_\omega^\pm \subset Z^\pm$  determined by  $\Delta_\omega \cap \Delta_{\omega'}$  coincide with  $Z_\omega^\pm \cap Z_{\omega'}^\pm \subset Z^\pm$  so that  $\Delta^\pm = \{\Delta_\omega^\pm\}_{\omega \in \Omega}$  are *geometric simplicial complexes* [33, Sec. 2.2.1]. With these assumptions in place, the correspondence between  $Z^-$  and  $Z^+$  determined by the triangulation  $(Z^-, Z^+, \Delta)$  uniquely defines a piecewise-affine function  $P : |\Delta^-| \rightarrow |\Delta^+|$  using the construction from [33, Sec. 3.1] where  $|\Delta^-| = \cup_{\omega \in \Omega} \Delta_\omega^- \subset \mathbb{R}^d, |\Delta^+| = \cup_{\omega \in \Omega} \Delta_\omega^+ \subset \mathbb{R}^c$  are termed the *carriers* [12, Sec. 2.2.1] of the geometric simplicial complexes  $\Delta^\pm$ .

**4.3 Piecewise-Linear Functions.** If a piecewise-affine function  $P : \mathbb{R}^d \rightarrow \mathbb{R}^c$  is *positively homogeneous*, that is,

$$\forall \alpha \geq 0, v \in \mathbb{R}^d : P(\alpha \cdot v) = \alpha \cdot P(v) \quad (21)$$

then  $P$  is *piecewise linear* [12, Prop. 2.2.1]. In this case,  $P$  admits a *conical subdivision* [12, Prop. 2.2.3], that is, there exists a finite collection  $\Sigma = \{\Sigma_\omega\}_{\omega \in \Omega}$  such that: (i)  $\Sigma_\omega \subset \mathbb{R}^d$  is a  $d$ -dimensional *polyhedral cone* for each  $\omega \in \Omega$ ;<sup>7</sup> (ii) the  $\Sigma_\omega$ 's cover  $\mathbb{R}^d$ ,<sup>8</sup> and (iii) the intersection  $\Sigma_\omega \cap \Sigma_{\omega'}$  is either empty or a *proper face* of both polyhedral cones for each  $\omega, \omega' \in \Omega$ .<sup>9</sup>

**4.4 Piecewise-Differentiable ( $PC^r$ ) Functions.** (This section is largely repeated from [8, Sec. 3.2].) The notion of piecewise-differentiability we employ was originally introduced in Ref. [35]; since the monograph [12] provides a more recent and comprehensive exposition, we adopt the notational conventions therein. Let  $r \in \mathbb{N} \cup \{\infty\}$  and  $D \subset \mathbb{R}^d$  be open. A continuous function  $f : D \rightarrow \mathbb{R}^c$  is called *piecewise  $C^r$*  if for every  $x_0 \in D$  there exists an open set  $U \subset D$  containing  $x_0$  and a finite collection  $\{f_j : U \rightarrow \mathbb{R}^c\}_{j \in J}$  of  $C^r$  functions such that for all  $x \in U$  we have  $f(x) \in \{f_j(x)\}_{j \in J}$ . The functions  $\{f_j\}_{j \in J}$  are called *selection functions* for  $f|_U$ , and is said to be a *continuous selection* of  $\{f_j\}_{j \in J}$

<sup>6</sup>There are more general definitions of ([complete] semi-)simplicial complexes and the closely related concept of  $\Delta$ -complexes in the literature [34, Ch. 2.1], [33, App. A.3.1]. Since we employ these concepts primarily in service of parameterizing piecewise-affine functions as in Ref. [33, Sec. 3.1], we adopt the (relatively restrictive) definitions of *combinatorial* and *geometric* simplicial complexes from Ref. [33, Sec. 2.2.1] in what follows.

<sup>7</sup>i.e.,  $\Sigma_\omega = \{\sum_{j=1}^{\ell_\omega} \alpha_j v_j^\omega : \{\alpha_j\}_{j=1}^{\ell_\omega} \subset [0, \infty)\}$ , some  $\{v_j\}_{j=1}^{\ell_\omega} \subset \mathbb{R}^d$  [12, Thm. 2.1.1], and  $\dim \Sigma_\omega = d$ .

<sup>8</sup>i.e.,  $\cup_{\omega \in \Omega} \Sigma_\omega = \mathbb{R}^d$ .

<sup>9</sup>i.e.,  $\Sigma_\omega \cap \Sigma_{\omega'} = \{\sum_{j=1}^{\ell_{\omega, \omega'}} \alpha_j v_j^{\omega, \omega'} : \{\alpha_j\}_{j=1}^{\ell_{\omega, \omega'}} \subset [0, \infty)\}$ , some  $\{v_j^{\omega, \omega'}\}_{j=1}^{\ell_{\omega, \omega'}} \subset \{v_j^\omega\}_{j=1}^{\ell_\omega} \cup \{v_j^{\omega'}\}_{j=1}^{\ell_{\omega'}}$ .

<sup>5</sup><https://github.com/gcouncil-cmu/bderv>

on  $U$ . A selection function  $f_j$  is said to be *active* at  $x \in U$  if  $f(x) = f_j(x)$ . We let  $PC^r(D, \mathbb{R}^c)$  denote the set of piecewise- $C^r$  functions from  $D$  to  $\mathbb{R}^c$ . Note that  $PC^r$  is closed under composition. The definition of piecewise  $C^r$  may at first appear unrelated to the intuition that a function ought to be piecewise-differentiable precisely if its “domain can be partitioned locally into a finite number of regions relative to which smoothness holds” [36, Sec. 1]. However, as shown in [36, Thm. 2], piecewise- $C^r$  functions are always piecewise differentiable in this intuitive sense.

Piecewise-differentiable functions possess a first-order approximation  $Df : TD \rightarrow T\mathbb{R}^c$  called the *Bouligand derivative* (or B-derivative) [12, Ch. 3]; this is the content of Ref. [12, Lemma 4.1.3]. Significantly, this B-derivative obeys generalizations of many techniques familiar from calculus, including the chain rule [12, Thm 3.1.1], fundamental theorem of calculus [12, Prop. 3.1.1], and implicit function theorem [37, Cor. 20]. We let  $Df(x; \delta x)$  denote the B-derivative of  $f$  evaluated on the tangent vector  $\delta x \in T_x D$ . The B-derivative is positively homogeneous, i.e.,  $\forall \delta x \in T_x D, \lambda \geq 0 : Df(x; \lambda \delta x) = \lambda Df(x; \delta x)$ , and coincides with the directional derivative of  $f$  in the  $\delta x \in T_x D$  direction. In addition, the B-derivative  $Df(x) : T_x D \rightarrow T_{f(x)} \mathbb{R}^c$  of  $f$  at  $x \in D$  is a continuous selection of the derivatives of the selection functions active at  $x$  [12, Prop. 4.1.3]

$$\forall \delta x \in T_x D : Df(x; \delta x) \in \{Df_j(x) \cdot \delta x\}_{j \in \mathcal{F}} \quad (22)$$

However, the function  $Df$  is generally *not* continuous at  $(x, \delta x) \in TD$ ; if it is, then  $f$  is  $C^1$  at  $x$  [12, Prop. 3.1.2].

#### 4.5 Event-Selected $C^r$ ( $EC^r$ ) Vector Fields and $PC^r$ Flows.

Vector fields with discontinuous right-hand sides and their associated flows have been studied extensively [38]. In [8, Defs. 1, 2], a special class of so-called event-selected  $C^r$  ( $EC^r$ ) vector fields were defined, which are allowed to be discontinuous along a finite number of codimension-1 submanifolds but do not exhibit *sliding* [10] along these submanifolds, and are  $C^r$  elsewhere. Importantly, as shown in Ref. [8, Thm. 5], an event-selected  $C^r$  vector field  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  generates a piecewise-differentiable flow, that is, there exists a function  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$  that is *piecewise  $C^r$*  ( $\phi \in PC^r$ ) in the sense defined in Ref. [12, Sec. 4.1] (summarized in Sec. 4.4) where  $\mathcal{F} \subset \mathbb{R} \times \mathbb{R}^d$  and

$$\forall (t, x) \in \mathcal{F} : \phi(t, x) = x + \int_0^t F(\phi(s, x)) ds \quad (23)$$

Since  $\phi$  is  $PC^r$ , it admits a first-order approximation  $D\phi : T\mathcal{F} \rightarrow T\mathbb{R}^d$  termed the *Bouligand* (or B-)derivative [12, Sec. 3.1], which is a continuous piecewise-linear function of tangent vectors at every  $(t, x) \in \mathcal{F}$ , that is, the directional derivative  $D\phi(t, x) : T_{(t,x)} \mathcal{F} \rightarrow T_{\phi(t,x)} \mathbb{R}^d$  is continuous and piecewise linear for all  $(t, x) \in \mathcal{F}$ .

#### 4.6 B-Derivative of an $EC^r$ Vector Field's $PC^r$ Flow.

Suppose  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  is an  $EC^r$  vector field with  $PC^r$  flow  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$ . Given a tangent vector  $(\delta t, \delta x) \in T_{(t,x)} \mathcal{F}$ , it was shown in Ref. [8, Sec. 7.1.4] that the value of the B-derivative  $D\phi(t, x; \delta t, \delta x) \in T_{\phi(t,x)} \mathbb{R}^d$  can be obtained by solving a jump-linear-time-varying differential equation [8, Eq. (70)], where the “jump” arises from a matrix  $\Xi_\omega$  determined by the sequence  $\omega$  in which the perturbed initial state  $x + \alpha \delta x$  crosses the surfaces of discontinuity of the vector field  $F$  for small  $\alpha > 0$  [8, Eq. (67)]. However, Ref. [8] did not provide a representation of the piecewise-linear operator  $D\phi(t, x)$  (and, to the best of our knowledge, neither has subsequent work). The key theoretical contribution of this paper, obtained in Sec. 5, is a representation of the B-derivative with respect to state,  $D_x \phi(t, x)$ , using a *triangulation* of its domain and codomain as defined in Ref. [33, Sec. 3.1] (and recalled in Sec. 4.2).

To inform the triangulation of the B-derivative  $D_x \phi(t, x)$ , we recall the values it takes on. Since the flow  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$  is

piecewise  $C^r$  ( $PC^r$ ), it is a *continuous selection* of a finite collection of  $C^r$  functions  $\{\phi_\omega : \mathcal{F}_\omega \rightarrow \mathbb{R}^d\}_{\omega \in \Omega}$  near  $(t, x) \in \mathcal{F}$ , where  $\mathcal{F}_\omega \subset \mathcal{F}$  is an open set containing  $(t, x)$  for each  $\omega \in \Omega$  [12, Sec. 4.1], and the B-derivative  $D_x \phi(t, x)$  is a continuous selection of the classical (*Fréchet* or *F*-)derivatives  $\{D_x \phi_\omega(t, x)\}_{\omega \in \Omega}$  [12, Prop. 4.1.3], that is,

$$\forall \delta x \in W_\omega \subset T_x \mathbb{R}^d : D_x \phi(t, x; \delta x) = D_x \phi_\omega(t, x) \cdot \delta x \quad (24)$$

where  $W_\omega \subset T_x \mathbb{R}^d$  is the subset of tangent vectors where the selection function  $D_x \phi_\omega$  is *essentially active* [12, Prop. 4.1.1]. If  $s, t \in \mathbb{R}$  and  $x \in \mathbb{R}^d$  are such that  $0 < s < t$  and the vector field  $F$  is  $C^r$  on  $\phi([0, t] \setminus \{s\}, x)$ , i.e., the trajectory initialized at  $x \in \mathbb{R}^d$  encounters exactly one discontinuity of  $F$  at  $\rho = \phi(s, x)$  on the time interval  $[0, t]$ , then  $D_x \phi_\omega(t, x)$  has the form

$$D_x \phi_\omega(t, x) = D_x \phi(t - s, \rho) \cdot [F_{+1}(\rho) \quad I_d] \cdot \Xi_\omega \cdot \begin{bmatrix} 0_d^\top \\ I_d \end{bmatrix} \cdot D_x \phi(s, x) \quad (25)$$

where  $F_{+1}$  is the  $C^r$  extension of  $F|_{\text{Int}D_{+1}}$  that exists by virtue of condition 2 in Def. 1 and  $\Xi_\omega \in \mathbb{R}^{(d+1) \times (d+1)}$  is the matrix from Ref. [8, Eq. (67)] corresponding to the selection function index  $\omega \in \Omega$ . In what follows, we will work in circumstances where the selection functions are indexed by the symmetric permutation group over  $n$  elements, i.e.,  $\Omega = S_n$ , and combine Eqs. (24) and (25) as

$$\forall \delta x \in W_\sigma \subset T_x \mathbb{R}^d : D_x \phi(t, x; \delta x) = D_x \phi(t - s, \rho) \cdot M_\sigma \cdot D_x \phi(s, x) \cdot \delta x \quad (26)$$

where the *saltation matrix*<sup>10</sup>  $M_\sigma \in \mathbb{R}^{d \times d}$  corresponding to index  $\sigma$  is defined by

$$M_\sigma = [F_{+1}(\rho) \quad I_d] \cdot \Xi_\sigma \cdot \begin{bmatrix} 0_d^\top \\ I_d \end{bmatrix} \quad (27)$$

**4.7 Local Approximation of an  $EC^r$  Vector Field.** Suppose vector field  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  is event-selected  $C^r$  with respect to  $h \in C^r(U, \mathbb{R}^n)$  at  $\rho \in U \subset \mathbb{R}^d$ . For  $b \in B_n = \{-1, +1\}^n$  let

$$\tilde{D}_b = \{x \in \mathbb{R}^d : b_j Dh_j(\rho)(x - \rho) \geq 0\} \quad (28)$$

and consider piecewise-constant vector field  $\tilde{F} : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  defined by

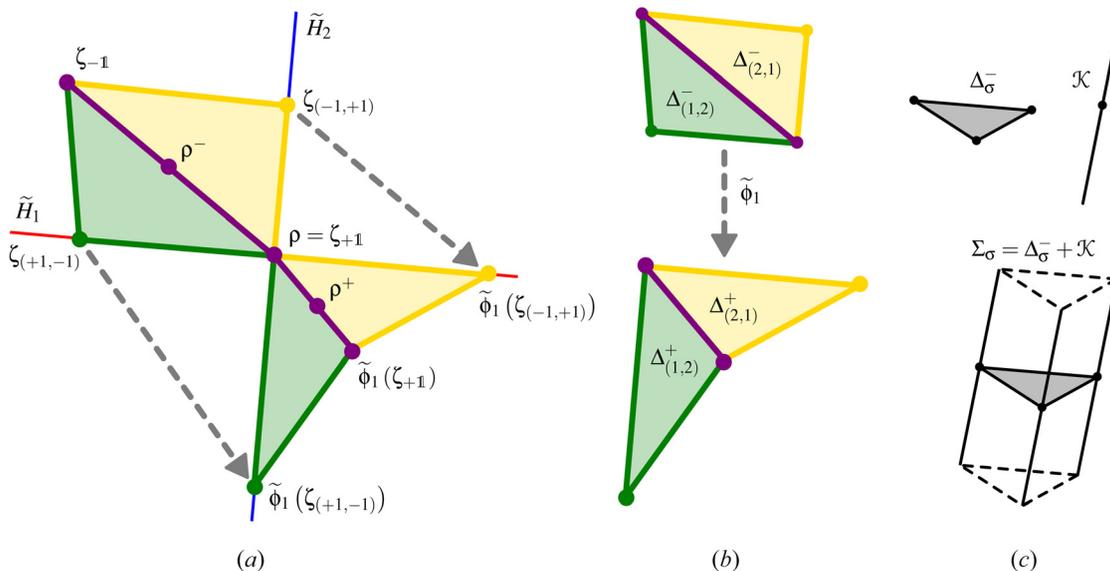
$$\forall b \in B_n, x \in \tilde{D}_b : \tilde{F}(x) = F_b(\rho) \quad (29)$$

where  $F_b$  is the  $C^r$  extension of  $F|_{\text{Int}D_b}$  that exists by virtue of condition 2 in Def. 1. Note that  $\tilde{F}$  is event-selected  $C^r$  with respect to the affine function  $h$  defined by

$$\forall x \in \mathbb{R}^d : \tilde{h}(x) = Dh(\rho)(x - \rho) \quad (30)$$

whence it generates a piecewise-differentiable flow  $\tilde{\phi} : \tilde{\mathcal{F}} \rightarrow \mathbb{R}^d$  where  $\tilde{\mathcal{F}} = \mathbb{R} \times \mathbb{R}^d$ . In Ref. [8, Sec. 7.1.3],  $\tilde{F}$  was referred to as the *sampled* vector field since it is obtained by “sampling” the selection functions  $F_b$  that define  $F$  near  $\rho$ , and it was noted that the function  $\tilde{\phi}$  is piecewise affine and it approximates the original vector field's flow  $\phi$  near  $\rho$ . We will leverage the algebraic properties of  $\tilde{\phi}$  and its relationship to  $\phi$  in what follows to obtain our results.

<sup>10</sup> $\Xi_\sigma \in \mathbb{R}^{(d+1) \times (d+1)}$  is referred to as a saltation matrix in Ref. [8, Sec. 7.1.4], but this usage is inconsistent with the original definition in Ref. [13].



**Fig. 5** Triangulation of time-1 flow  $\tilde{\phi}_1$  of sampled system associated with planar  $EC^r$  vector field. (a) For each  $b \in \{-1, +1\}^2$ , the point  $\zeta_b$  defined by Eq. (33) flows from  $\tilde{D}_{-1}$  to  $\tilde{D}_{+1}$  in 1 (one) unit of time via the sampled system illustrated in Fig. 1(b) and defined in Sec. 4.7. (b) The sets  $\{\zeta_{-1}, \zeta_{+1}, \zeta_{(+1,-1)}\}, \{\zeta_{-1}, \zeta_{+1}, \zeta_{(-1,+1)}\}$  indexed by Eq. (37) define geometric simplices  $\Delta_{(1,2)}^-, \Delta_{(2,1)}^+$  that pass through subspaces  $\tilde{H}_1, \tilde{H}_2$  in the same order. (c) For each  $\sigma \in \{(1, 2), (2, 1)\}$ , the set  $\Sigma_\sigma$  is a direct sum of  $\Delta_\sigma^-$  with subspace  $\mathcal{K}$ .

**4.8 Time-to-Impact for an  $EC^r$  Vector Field and Its Local Approximation.** Suppose vector field  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  is event-selected  $C^r$  with respect to  $h \in C^r(U, \mathbb{R}^n)$  at  $\rho \in U \subset \mathbb{R}^d$ , and let  $\phi \in PC^r(\mathcal{F}, \mathbb{R}^d)$  be its piecewise-differentiable flow. Then Ref. [8, Thm. 7] ensures there exists a piecewise-differentiable time-to-impact function  $\tau \in PC^r(U, \mathbb{R}^n)$  such that,  $\forall x \in U, j \in \{1, \dots, n\}$

$$\phi(\tau_j(x), x) \in H_j = h_j^{-1}(h_j(\rho)) \quad (31)$$

i.e.,  $x$  flows to the surface  $H_j$  in time  $\tau_j(x)$ . Similarly, applying [8, Thm. 7] to the sampled vector field  $\tilde{F} : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  and piecewise-affine flow  $\tilde{\phi} : \tilde{\mathcal{F}} \rightarrow \mathbb{R}^d$  associated with  $F$  at  $\rho$  constructed in Sec. 4.7 ensures there exists a piecewise-affine time-to-impact function  $\tilde{\tau} : \mathbb{R}^d \rightarrow \mathbb{R}^n$  such that  $\forall x \in \mathbb{R}^d, j \in \{1, \dots, n\}$

$$\tilde{\phi}(\tilde{\tau}_j(x), x) \in \tilde{H}_j = \rho + \ker Dh_j(\rho) \quad (32)$$

i.e., the point  $x$  flows to the affine subspace  $\tilde{H}_j$  in time  $\tilde{\tau}_j(x)$ .

## 5 Representation

Our main theoretical result is an explicit representation for the Bouligand (or B-)derivative of the piecewise-differentiable flow generated by an event-selected  $C^r$  vector field. To that end, let  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  be an event-selected  $C^r$  vector field and  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$  its piecewise-differentiable flow. In what follows, we will assume that  $s, t \in \mathbb{R}$  and  $x \in \mathbb{R}^d$  are such that  $0 < s < t$  and the vector field  $F$  is  $C^r$  on  $\phi([0, t] \setminus \{s\}, x)$ . Although a general trajectory can encounter more than one point of discontinuity for  $F$ , such points are isolated [8, Lem. 6], so the Chain Rule for B-differentiable functions [12, Thm. 3.1.1] can be applied to triangulate the desired flow derivative by composing the triangulated flow derivatives associated with each point. Thus, without loss of generality, we restrict our attention to portions of trajectories that encounter one point of discontinuity for  $F$ , which point lies at the intersection of  $n$  surfaces of discontinuity for  $F$ . We assume  $n > 1$  because at least two surfaces are needed for our results to be useful: when  $n = 1$  the desired B-derivative is linear [13], so it may be represented and employed in computations as a matrix.

The B-derivative  $D_x \phi(t, x) : T_x \mathbb{R}^d \rightarrow T_{\phi(t,x)} \mathbb{R}^d$  we seek is a continuous piecewise-linear function, so it can be parsimoniously

represented using a triangulation [33, Sec. 3.1], that is, a combinatorial simplicial complex (as defined in Sec. 4.2) each of whose vertices are associated with a pair of (tangent) vectors—one each in the domain and codomain of  $D_x \phi(t, x)$ . We will obtain this triangulation via an indirect route: in Sec. 5.1, we triangulate the piecewise-affine flow  $\tilde{\phi}$  introduced in Sec. 4.7; in Sec. 5.2, we differentiate our representation of  $\tilde{\phi}$  to obtain a triangulation of the B-derivative  $D_x \tilde{\phi}$ ; in Sec. 5.3, we show how the B-derivative  $D_x \phi$  can be obtained from  $D_x \tilde{\phi}$ , providing a triangulation of the desired derivative.

**5.1 Triangulation.** The goal of this section is to triangulate the piecewise-affine flow  $\tilde{\phi}$  introduced in Sec. 4.7. To that end, let  $\rho = \phi(s, x)$  and suppose<sup>11</sup>  $\text{rank } Dh(\rho) = n$  so  $\{\delta\rho \in T_\rho \mathbb{R}^d : b = \text{sign } Dh(\rho) \cdot \delta\rho\}$  has nonempty interior for each  $b \in \{-1, +1\}^n = B_n$ . Letting  $\mathcal{K} = \ker Dh(\rho) \subset T_\rho \mathbb{R}^d$  denote the kernel of  $Dh(\rho)$  and  $\mathcal{K}^\perp$  its orthogonal complement, for each  $b \in B_n$  there exists a unique<sup>12,13</sup>  $\zeta_b \in \mathcal{K}^\perp + \{\rho\}$  such that

$$Dh_{b>0}(\rho)(\zeta_b - \rho) = 0, \quad Dh_{b<0}(\rho)(\zeta_b + F_b(\rho) - \rho) = 0 \quad (33)$$

where  $h_{b>0}$  (respectively,  $h_{b<0}$ ) denotes the function obtained by selecting components  $h_j$  of  $h$  for which  $b_j = +1$  (respectively,  $b_j = -1$ ). The vectors defined by Eq. (33) have special significance for the piecewise-affine flow  $\tilde{\phi}$  introduced in Sec. 4.7 (see Fig. 5(a))

$$\forall b \in B_n : \zeta_b \in \tilde{D}_{-1}, \quad \tilde{\phi}(1, \zeta_b) = \zeta_b + F_b(\rho) \in \tilde{D}_{+1} \quad (34)$$

that is, the point  $\zeta_b$  lies “before” all event surface tangent planes and flows in 1 (one) unit of time to  $\zeta_b + F_b(\rho)$ , which lies “after” all event surface tangent planes (neither “before” nor “after” should be interpreted strictly). We denote the collections of these vectors as follows:

<sup>11</sup>As observed in Ref. [8, Sec. 7.1.5], first-order approximations of an  $EC^r$  vector field’s  $PC^r$  flow are not affected by flow between surfaces that are tangent at  $\rho$ , so we assume such redundancy has been removed.

<sup>12</sup>Here and in what follows we mildly abuse notation via the natural vector space isomorphism  $\mathbb{R}^d \simeq T_\rho \mathbb{R}^d$ .

<sup>13</sup> $\text{rank } Dh(\rho) = n$  ensures uniqueness since (i)  $\mathcal{K}^\perp$  is  $n$ -dimensional, (ii) the rows of  $Dh(\rho)$  are linearly independent, and hence (iii) there are  $n$  independent equations in the  $n$  unknowns needed to specify  $\zeta_b$  in Eq. (33).

$$Z^- = \{\zeta_b\}_{b \in B_n}, \quad Z^+ = \{\zeta_b + F_b(\rho)\}_{b \in B_n} \quad (35)$$

In what follows, it will be convenient to use an element  $\sigma \in S_n$  of the symmetric permutation group over  $n$  elements to specify  $n+1$  elements of  $b \in B_n$  as follows: for each  $k \in \{0, \dots, n\}$ , let  $\sigma(\{0, \dots, k\}) \subset \{1, \dots, n\}$  specify the unique  $b \in B_n$  whose  $j$ th component is  $+1$  if and only if  $j \in \sigma(\{0, \dots, k\})$ . Note that this identification yields, with some abuse of notation,  $\sigma(\{0\}) = -1$ ,  $\sigma(\{0, \dots, n\}) = +1$ . Finally, note that the following are linearly independent collections of vectors:

$$\{\zeta_{\sigma(\{0, \dots, k\})} - \rho\}_{k=0}^{n-1} \quad (36a)$$

$$\{\zeta_{\sigma(\{0, \dots, k\})} + F_{\sigma(\{0, \dots, k\})}(\rho) - \rho\}_{k=1}^n \quad (36b)$$

This fact is easily verified for Eq. (36a) in coordinates where  $Dh(\rho) = [I_n \ 0_{n \times (d-n)}]$ , where the fact follows for Eq. (36b) by Eqs. (36a) and (34) via [8, Cor. 5(c)] (time- $t$  flow of an  $EC^r$  vector field is a homeomorphism of the state space).

Let  $\Delta$  be the combinatorial simplicial complex over vertices  $B_n$  with maximal  $n$ -simplices indexed by  $\sigma \in S_n$  via

$$\Delta_\sigma = \{\sigma(\{0, \dots, k\})\}_{k=0}^n \in \Delta \quad (37)$$

where we regard  $\sigma(\{0, \dots, k\})$  as an element of  $B_n$  using the same abuse of notation employed in Eq. (36). By associating each vertex  $b \in B_n$  with the vector  $\zeta_b \in Z^- \subset \mathbb{R}^d$ , every  $n$ -simplex  $\Delta_\sigma$  determines an  $n$ -dimensional geometric simplex  $\Delta_\sigma^- \subset \mathbb{R}^d$ , the dimensionality of which is ensured by Eq. (36a); similarly, Eq. (36b) ensures that associating each  $b \in B_n$  with  $(\zeta_b + F_b(\rho)) \in Z^+ \subset \mathbb{R}^d$  determines an  $n$ -dimensional geometric simplex  $\Delta_\sigma^+ \subset \mathbb{R}^d$  from each  $n$ -simplex  $\Delta_\sigma$ . Refer to Fig. 5(b) for an illustration when  $n=2$ . The triple  $(Z^-, Z^+, \Delta)$  parameterizes a continuous piecewise-affine homeomorphism  $P: |\Delta^-| \rightarrow |\Delta^+|$  using the construction from Ref. [33, Sec. 3.1] (summarized in Sec. 4.2), where  $|\Delta^\pm| = \cup_{\sigma \in S_n} \Delta_\sigma^\pm \subset \mathbb{R}^d$  denote the carriers of the geometric simplicial complexes  $\Delta^\pm$ .

We now show that the piecewise-affine function  $P$  constructed above is the nonlinear part of the time-1 flow of the sampled system  $\phi_1$  restricted to  $|\Delta^-|$ . For each  $\sigma \in S_n$  we extend the  $n$ -dimensional geometric simplex  $\Delta_\sigma^-$  determined by the  $n$ -simplex  $\Delta_\sigma$  via direct sum with the  $(d-n)$ -dimensional subspace  $\mathcal{K}$  to obtain a  $d$ -dimensional polyhedron  $\Sigma_\sigma$  (see Fig. 5(c)), and let  $|\Sigma| = \cup_{\sigma \in S_n} \Sigma_\sigma$ . Note that  $\mathcal{K}$  is a subset of the lineality space of  $\Sigma_\sigma$  for each  $\sigma \in S_n$ .

LEMMA 1.  $\phi_1|_{|\Sigma|}$  is piecewise affine and

$$\forall z \in |\Delta^-|, \quad \xi \in \mathcal{K}: \tilde{\phi}_1(z + \xi) = P(z) + \xi \quad (38)$$

*Proof.* This proof will proceed in two steps: (i) show that  $\tilde{\phi}_1(z) = P(z)$  for all  $z \in |\Delta^-|$ ; (ii) show that  $\tilde{\phi}_1(z + \xi) = \phi_1(z) + \xi$  for all  $z \in |\Delta^-|, \xi \in \mathcal{K}$ .

(i) Recall from Eq. (34) that  $\tilde{\phi}_1|_{Z^-} = P|_{Z^-}$  where  $Z^-$  is the vertex set for the geometric simplicial complex  $\Delta^-$ . For each  $\sigma \in S_n$  let  $Z_\sigma = \{\zeta_b\}_{b \in \Delta_\sigma^-}$  denote the vertex set of the  $n$ -dimensional geometric simplex  $\Delta_\sigma^-$ . Then we claim that each  $z \in \Delta_\sigma^-$  passes through the same sequence of transition surfaces as each  $\zeta_b \in Z_\sigma$ . To verify this claim, we use the piecewise-affine time-to-impact function  $\tilde{\tau}: \mathbb{R}^d \rightarrow \mathbb{R}^n$  from Sec. 4.8. Note that  $\zeta_b$  impacts affine subspace  $\tilde{H}_j$  at time 1 if  $b_j = -1$  and at time 0 if  $b_j = +1$ , i.e.,

$$\tilde{\tau}_j(\zeta_b) = \begin{cases} 1, & b_j = -1 \\ 0, & b_j = +1 \end{cases} \quad (39)$$

Convex combination  $\alpha \zeta_b + (1-\alpha)\zeta_{b'}$ ,  $\alpha \in (0, 1)$ ,  $b, b' \in \Delta_\sigma$ , impacts  $\tilde{H}_j$  at time  $\tilde{\tau}_j(\alpha \zeta_b + (1-\alpha)\zeta_{b'})$  that is: 0 if  $b_j = b'_j = +1$ , 1 if  $b_j = b'_j = -1$ , and between 0 and 1 otherwise. More generally, any point  $z \in \Delta_\sigma^-$  is a convex combination of the vertices  $Z_\sigma$ , whence it impacts surfaces in the order prescribed by  $\sigma$ , so  $\forall z \in \Delta_\sigma^-$

$$0 \leq \tilde{\tau}_{\sigma(1)}(z) \leq \tilde{\tau}_{\sigma(2)}(z) \leq \dots \leq \tilde{\tau}_{\sigma(n)}(z) < 1 \quad (40)$$

Thus,  $\tilde{\phi}_1|_{\Delta_\sigma^-}$  is affine and agrees with  $P|_{\Delta_\sigma^-}$ . Since  $|\Delta^-| = \cup_{\sigma \in S_n} \Delta_\sigma^-$ , we have  $\tilde{\phi}_1|_{|\Delta^-|} = P$ .

(ii) We now show that the piecewise-affine map  $\tilde{\phi}_1$  is indifferent to every  $\xi \in \mathcal{K} = \ker Dh(\rho)$ , so for every  $z \in |\Delta^-|$

$$\tilde{\phi}_1(z + \xi) = \tilde{\phi}_1(\rho + (z + \xi - \rho)) \quad (41a)$$

$$= \tilde{\phi}_1(\rho) + D\tilde{\phi}_1(\rho; z + \xi - \rho) \quad (41b)$$

$$= \tilde{\phi}_1(\rho) + D\tilde{\phi}_1(\rho; z - \rho) + \xi \quad (41c)$$

$$= \tilde{\phi}_1(z) + \xi \quad (41d)$$

Indeed, Eq. (41a) since  $z + \xi = \rho + (z + \xi - \rho)$ ; Eq. (41b) since  $\tilde{\phi}_1$  is affine on the segment  $\{\rho + \alpha(z + \xi - \rho) : \alpha \in [0, 1]\}$ ; Eq. (41c) since each piece of the continuous piecewise-linear B-derivative  $D\tilde{\phi}_1(\rho)$  is specified by a saltation matrix (as recalled in Sec. 4.4) that is the product of matrices of the form  $(I_d + g \cdot Dh_j(\rho))$  [8, Eq. (60)], thus  $\xi \in \mathcal{K} = \ker Dh(\rho)$  is transformed by  $I_d$ ; Eq. (41d) for the same reason as Eq. (41b). ■

**5.2 B-Derivative of  $\tilde{\phi}_1$ .** The goal of this section is to differentiate the representation of  $\tilde{\phi}$  from Sec. 5.1 to obtain a triangulation of the B-derivative  $D\tilde{\phi}_1: T_{\rho^-} \mathbb{R}^d \rightarrow T_{\rho^+} \mathbb{R}^d$  between the following two points:

$$\rho^- = \rho - \frac{1}{2}F_{-1}(\rho), \quad \rho^+ = \tilde{\phi}(1, \rho^-) = \rho + \frac{1}{2}F_{+1}(\rho) \quad (42)$$

LEMMA 2.  $B = D\tilde{\phi}_1(\rho^-): T_{\rho^-} \mathbb{R}^d \rightarrow T_{\rho^+} \mathbb{R}^d$  satisfies:

1.  $B$  specifies how  $\tilde{\phi}_1$  varies relative to  $\tilde{\phi}_1(\rho^-)$

$$\forall x \in |\Sigma|: \tilde{\phi}_1(x) = \tilde{\phi}_1(\rho^-) + B(x - \rho^-) \quad (43)$$

2.  $B$  is piecewise linear with conical subdivision

$$\Sigma' = \{\Sigma'_\sigma = \text{cone}(\Sigma_\sigma - \rho^-) : \sigma \in S_n\} \quad (44)$$

3.  $B|_{\Sigma'_\sigma}$  is linear for all  $\sigma \in S_n$  and  $\forall \delta \rho \in \Sigma'_\sigma$ :

$$B(\delta \rho) = M_\sigma \cdot \delta \rho \quad (45)$$

4.  $\mathcal{L} = \mathcal{K} + \text{span } F_{-1}(\rho)$  is a  $(d-n+1)$ -dimensional lineality space for  $\Sigma'$  and  $\forall \sigma \in S_n$

$$\Sigma'_\sigma = \mathcal{L} + \text{cone}\left\{\Pi_{\mathcal{L}}^\perp \cdot (\zeta_{\sigma(\{0, \dots, k\})} - \rho)\right\}_{k=1}^{n-1} \quad (46)$$

where  $\Pi_{\mathcal{L}}^\perp$  is the orthogonal projection onto  $\mathcal{L}^\perp$ ;

5.  $B|_{\mathcal{L}}$  is linear and  $\forall \delta \rho \in T_{\rho^-} \mathbb{R}^d$

$$B(\delta \rho) = B(\Pi_{\mathcal{L}} \cdot \delta \rho) + B(\Pi_{\mathcal{L}}^\perp \cdot \delta \rho) \quad (47)$$

where  $\Pi_{\mathcal{L}}$  is the orthogonal projection onto  $\mathcal{L}$ .

*Proof.* Each point follows from straightforward application of results in Ref. [12]: (1.), (2.), and (3.) are conclusions (4.), (3.), and (2.), respectively, of Ref. [12, Prop. 2.2.6]; (4.) follows from the definitions of lineality space [12, Sec. 2.1.2] and the  $\zeta_b$ 's Eq. (33); (5.) is a restatement of Ref. [12, Lem. 2.3.2]. ■

**5.3 B-Derivative of  $\phi$**  The goal of this section is to show that the piecewise-linear function  $B$  triangulated in Sec. 5.2 gives the nonlinear part of the desired B-derivative  $D_x \phi(t, x)$  and<sup>14</sup>

<sup>14</sup>Here and in what follows we mildly abuse notation via the natural vector space isomorphisms  $\mathbb{R}^d \simeq T_{\rho^-} \mathbb{R}^d \simeq T_{\rho^+} \mathbb{R}^d \simeq T_{\rho} \mathbb{R}^d$ .

$$W_\sigma = D_x \phi(s, x)^{-1} (\Sigma'_\sigma) \subset T_x \mathbb{R}^d \quad (48)$$

is the cone of tangent vectors where the saltation matrix  $M_\sigma$  is active in Eq. (26).

**THEOREM 1.** *Suppose vector field  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  is event-selected  $C^r$  with respect to  $h : \mathbb{R}^d \rightarrow \mathbb{R}^n$  at  $\rho$ . Let  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$  be the  $PC^r$  flow of  $F$  and  $s, t \in \mathbb{R}, x \in \mathbb{R}^d$  be such that  $0 < s < t$  and  $F$  is  $C^r$  on  $\phi([0, t] \setminus \{s\}, x) \subset \mathbb{R}^d$ . Then with  $\rho = \phi(s, x)$ , the B-derivative of the flow  $\phi$  with respect to state,  $D_x \phi(t, x) : T_x \mathbb{R}^d \rightarrow T_{\phi(t, x)} \mathbb{R}^d$ , is given  $\forall \delta x \in W_\sigma \subset T_x \mathbb{R}^d$  by*

$$D_x \phi(t, x; \delta x) = D_x \phi(t - s, \rho) \cdot B(D_x \phi(s, x) \cdot \delta x), \quad (49a)$$

$$D_x \phi(t, x; \delta x) = D_x \phi(t - s, \rho) \cdot M_\sigma \cdot D_x \phi(s, x) \cdot \delta x \quad (49b)$$

where  $B$  is the continuous piecewise-linear function from Lemma 2,  $W_\sigma$  is the cone from Eq. (48), and  $M_\sigma$  is the saltation matrix from Eq. (27).

*Proof.* Note that Eq. (49a) follows from Eq. (49b) by Eq. (45), and the fact that “pieces” of the B-derivative  $D_x \phi(t, x)$  are determined by the collection of saltation matrices  $\{M_\sigma\}_{\sigma \in S_n}$  was recalled in Sec. 4.4. Thus, to establish Eq. (49b) what remains to be shown is that  $M_\sigma$  is the active “piece” for all  $\delta x \in W_\sigma$ , i.e., that  $\{W_\sigma\}_{\sigma \in S_n}$  is a conical subdivision for the piecewise-linear operator  $D_x \phi(t, x)$ , with  $W_\sigma$  as defined in Eq. (48).

Given  $\delta x \in \text{Int } W_\sigma$  let  $\delta \rho = D_x \phi(s, x) \cdot \delta x \in \text{Int } \Sigma'_\sigma$  so that

$$\tilde{\tau}_{\sigma(1)}(\rho + \delta \rho) < \tilde{\tau}_{\sigma(2)}(\rho + \delta \rho) < \dots < \tilde{\tau}_{\sigma(n)}(\rho + \delta \rho) \quad (50)$$

where  $\tilde{\tau}$  is the time-to-impact function for the sampled system as defined in Eq. (32). Note that  $D_x \phi(t, x)$  is linear on  $\text{span } F(x)$ , i.e.,  $\forall \alpha \in \mathbb{R}$

$$D_x \phi(t, x; \delta x + \alpha F(x)) = D_x \phi(t, x; \delta x) + \alpha F(\phi(t, x)) \quad (51)$$

so without loss of generality, we assume  $\delta \rho \in \text{Int } \tilde{D}_{-1}$  by translating  $\delta x$  in the  $-F(x)$  direction. We claim for all  $\alpha > 0$  sufficiently small that  $\phi(t, x + \alpha \delta x)$  passes through the event surfaces with the same sequence as  $\phi(1, \rho + \alpha \delta \rho)$ , i.e., that

$$\tau_{\sigma(1)}(x + \alpha \delta x) < \tau_{\sigma(2)}(x + \alpha \delta x) < \dots < \tau_{\sigma(n)}(x + \alpha \delta x) \quad (52)$$

where  $\tau$  is the time-to-impact function defined in Eq. (31). To see this, note that  $\forall k \in \{1, \dots, n\}$

$$\tau_{\sigma(k)}(x + \alpha \delta x) - \tau_{\sigma(k)}(x) \quad (53a)$$

$$= D\tau_{\sigma(k)}(x; \alpha \delta x) + O(\alpha^2) \quad (53b)$$

$$= D\tilde{\tau}_{\sigma(k)}(\rho; \alpha \delta \rho) + O(\alpha^2) \quad (53c)$$

$$= \tilde{\tau}_{\sigma(k)}(\rho + \alpha \delta \rho) - \tilde{\tau}_{\sigma(k)}(\rho) + O(\alpha^2) \quad (53d)$$

where: Eq. (53b) since  $\tau$  is  $PC^r$ ; Eq. (53c) since  $\delta \rho = D_x \phi(s, x) \cdot \delta x$  and  $D\tau(x; \delta x), D\tilde{\tau}(\rho; \delta \rho)$  are determined by the same data, namely,  $Dh_{\sigma(k)}(\rho)$  and  $F_{-1}(\rho)$ ; Eq. (53d) since  $\delta \rho \in \Sigma'_\sigma$ . Combining the approximation Eq. (53) with Eq. (50) yields Eq. (52) as desired.

We conclude that  $\{W_\sigma\}_{\sigma \in S_n}$  is a conical subdivision for the piecewise-linear operator  $D_x \phi(t, x)$ , which verifies Eq. (49). ■

*Remark 1.* The only nonclassical part of the B-derivative of the flow in Eq. (49a) is the piecewise-linear function  $B$ . Although there are  $n!$  pieces of  $B$  in general, we explicitly represent all pieces using a triangulation of  $2^n$  sample points defined in Eq. (35), achieving a substantial reduction—from factorial to “merely” exponential—of the information needed to represent the first-order approximation of the flow. Note that  $B$  implicitly determines the transition sequence  $\sigma$  associated with the perturbation direction  $\delta x$  in Eq. (49a), whereas

this sequence must be explicitly specified to select the appropriate saltation matrix  $M_\sigma$  in Eq. (49b).

## 6 Computation

We now attend to the complexity of the computational tasks required to construct or evaluate the B-derivative representation from Sec. 5. To that end, let  $F : \mathbb{R}^d \rightarrow T\mathbb{R}^d$  be an event-selected  $C^r$  vector field with respect to  $h \in C^r(\mathbb{R}^d, \mathbb{R}^n)$  and  $\phi : \mathcal{F} \rightarrow \mathbb{R}^d$  its piecewise- $C^r$  flow, and assume  $s, t \in \mathbb{R}$  and  $x \in \mathbb{R}^d$  are such that  $0 < s < t$ ,  $\rho = \phi(s, x)$ , and the vector field  $F$  is  $C^r$  on  $\phi([0, t] \setminus \{s\}, x)$ .

We seek to compute  $D_x \phi(t, x; \delta x)$  given  $\delta x \in T_x \mathbb{R}^d$ . Since Eq. (49a) from Theorem 1 yields

$$D_x \phi(t, x; \delta x) = D_x \phi(t - s, x) \cdot B(D_x \phi(s, x) \cdot \delta x) \quad (54)$$

where  $B : T_\rho \mathbb{R}^d \rightarrow T_\rho \mathbb{R}^d$ , the crux of the computation is

$$\delta \rho^+ = B(\delta \rho^-) \quad (55)$$

where  $\delta \rho^- = D_x \phi(s, x) \cdot \delta x$ . In fact, Lemma 2 offers further simplification via Eq. (47): since  $B = B \circ \Pi_{\mathcal{L}} + B \circ \Pi_{\mathcal{L}^\perp}$  where  $B \circ \Pi_{\mathcal{L}}$  is the linear function

$$B \circ \Pi_{\mathcal{L}} \cdot \delta \rho^- = \left( I_d + (F_{+1}(\rho) - F_{-1}(\rho)) \cdot \frac{F_{-1}(\rho)^\top}{\|F_{-1}(\rho)\|^2} \right) \cdot \Pi_{\mathcal{L}} \cdot \delta \rho^- \quad (56)$$

only the piecewise-linear function  $B \circ \Pi_{\mathcal{L}^\perp}$  (equivalently, the restriction  $B|_{\mathcal{L}^\perp}$ ) requires special consideration. In what follows, we will assume the following data, needed to construct the *sampled system* illustrated in Fig. 1(b), are given: linearly independent normal vectors for the surfaces of discontinuity, i.e.,  $Dh(\rho) \in \mathbb{R}^{n \times d}$  with  $\text{rank } Dh(\rho) = n$ ; limiting values of the vector field at the point of intersection, i.e.,  $F_b(\rho) \in T_\rho \mathbb{R}^d$  for each  $b \in B_n$ ; and F-derivatives of the continuously differentiable parts of the flow, i.e.,  $D_x \phi(s, x), D_x \phi(t - s, x) \in \mathbb{R}^{d \times d}$ .

**6.1 Constructing the B-Derivative.** Lemma 2 demonstrates that there are  $n!$  pieces of the piecewise-linear function  $B$ , namely, the collection of saltation matrices  $\{M_\sigma\}_{\sigma \in S_n}$  in Eq. (45) that are active on the corresponding polyhedral cones in the conical subdivision  $\Sigma' = \{\Sigma'_\sigma\}_{\sigma \in S_n}$  in Eq. (44). These polyhedral cones are generated by the  $2^{n-1}$  points  $\{\zeta_b : b \in B_n \setminus \{-1, +1\}\}$  in Eq. (46). For each  $b \in B_n$ , the point  $\zeta_b \in \mathcal{K}^\perp + \{\rho\}$  where  $\mathcal{K} = \ker Dh(\rho)$  can be determined by solving the  $n$  affine equations with  $n$  unknowns in Eq. (33). Given  $\sigma \in S_n$ , the linear piece  $B|_{\mathcal{L}^\perp \cap \Sigma'_\sigma}$  can be constructed using the *saltation matrix* [8, Sec. 7.1.6] since  $B(\delta \rho^-) = M_\sigma \cdot \delta \rho^-$  for all  $\delta \rho^- \in \mathcal{L}^\perp \cap \Sigma'_\sigma$  where<sup>15</sup>

$$M_\sigma = \prod_{k=0}^{n-1} \left( I_d + \frac{(F_{\sigma(0:k+1)}(\rho) - F_{\sigma(0:k)}(\rho))}{Dh_{\sigma(0:k)}(\rho) \cdot F_{\sigma(0:k)}(\rho)} \cdot Dh_{\sigma(0:k)}(\rho) \right) \quad (57)$$

or using *barycentric coordinates* [33, Eq. (3.1)] since  $B(\delta \rho^-) = Z_\sigma^+ \cdot (Z_\sigma^-)^{-1} \cdot \delta \rho^- \in \mathcal{L}^\perp \cap \Sigma'_\sigma$  where

$$Z_\sigma^+ = [z_{\sigma(0:1)}^+ \ z_{\sigma(0:2)}^+ \ \dots \ z_{\sigma(0:n-1)}^+] \in \mathbb{R}^{d \times (n-1)} \quad (58)$$

$$\forall b \in \Delta'_\sigma : z_b^- = \Pi_{\mathcal{L}}^\perp \cdot (\zeta_b - \rho), \ z_b^+ = B|_{\mathcal{L}^\perp}(z_b^-) \quad (59)$$

<sup>15</sup>We mildly abuse notation as in Sec. 5.1 by using  $\sigma \in S_n$  to specify  $n+1$  elements of  $b \in B_n$ ; for each  $k \in \{0, \dots, n\}$ , we let  $\sigma(0:k) = \sigma(\{0, \dots, k\}) \subset \{1, \dots, n\}$  specify the unique  $b \in B_n$  whose  $j$ -th component is +1 if and only if  $j \in \sigma(\{0, \dots, k\})$ .

$$\Delta'_\sigma = \{\sigma(\{0, 1, \dots, k\})\}_{k=1}^{n-1} \quad (60)$$

note that the pseudo-inverse  $(Z_\sigma^-)^\dagger$  is injective on  $\mathcal{L}^\perp \cap \Sigma'_\sigma$  by Eqs. (36a) and (46). Although the matrices  $M_\sigma, Z_\sigma^+ \cdot (Z_\sigma^-)^\dagger \in \mathbb{R}^{d \times d}$  define the same linear transformation on the  $(n-1)$ -dimensional cone  $\mathcal{L}^\perp \cap \Sigma'_\sigma$ , they are generally not the same matrix. We conclude by noting that constructing the saltation matrix in Eq. (57) requires  $O(nd^2)$  time and  $O(d^2)$  space, whereas constructing the Barycentric coordinates in Eq. (58) requires  $O(n^2d^2)$  time and  $O(d^2)$  space (although evaluating the expression  $Z_\sigma^+ \cdot (Z_\sigma^-)^\dagger \cdot \delta\rho^-$  requires only  $O(nd^2)$  time given  $Z_\sigma^+$ ).

**6.2 Evaluating the B-Derivative.** One obvious strategy to evaluate  $B$  on  $\delta\rho^- \in T_\rho\mathbb{R}^d$  is to (i) determine  $\sigma \in S_n$  such that  $\delta\rho^- \in \Sigma'_\sigma$  then (ii) apply the corresponding saltation matrix or barycentric coordinates calculation from Sec. 6.1. The general formulation of (i), termed the *point location* problem in the computational geometry literature, is “essentially open” [39, Sec. 6.5]. For an arrangement of  $m$  hyperplanes in  $\mathbb{R}^d$ , queries can be answered in  $O(d \log m)$  time at the expense of  $O(m^d)$  space [40]. In our context, the conical subdivision  $\Sigma'$  in Eq. (46) is determined by an arrangement of  $m = O(n!^2)$  hyperplanes, so this general-purpose algorithm has time complexity  $O(d \log n!) = O(dn \log n)$  and space complexity  $O(n!^d)$ .

The relationship established by Eq. (43) between the desired B-derivative and the flow of the *sampled system* illustrated in Fig. 1(b) suggests a different strategy, summarized in Fig. 2, with slightly worse  $O(n^2d)$  time complexity but dramatically superior  $O(d)$  space complexity. To understand the strategy, interpret the tangent vector  $\delta\rho^- \in T_\rho\mathbb{R}^d$  as a perturbation away from the point  $\rho^- = \rho - \frac{1}{2}F_{-1}(\rho)$  that flows through  $\rho$  to  $\rho^+ = \rho + \frac{1}{2}F_{+1}(\rho)$  in one unit of time and observe that<sup>16</sup>  $\delta\rho^+ = \tilde{\phi}_1(\rho^- + \delta\rho^-) - \rho^+ = B(\delta\rho^-)$  as in Eq. (43). The flow of the sampled system  $\tilde{\phi}_1$  is piecewise affine, and can be evaluated on a given perturbation vector  $\delta\rho^-$  by performing a sequence of  $n$  affine projections (one for each of the affine subspaces  $\{\tilde{H}_j\}_{j=1}^n$  where  $\tilde{F}$  is discontinuous) specified by the permutation  $\sigma \in S_n$  for which  $\delta\rho^- \in \Sigma'_\sigma$ . Fortunately, the sequence  $\sigma$  can be determined inductively as follows: First, define

$$\begin{aligned} \delta t_1 &= 0 \\ \delta\rho_1 &= \delta\rho^- \\ \sigma(1) &= \operatorname{argmin} \left\{ -\frac{Dh_j(\rho) \cdot \delta\rho_1}{Dh_j(\rho) \cdot F_{-1}(\rho)} : j \in \{1, \dots, n\} \right\} \\ \tau_1 &= -\frac{Dh_{\sigma(1)}(\rho) \cdot \delta\rho_1}{Dh_{\sigma(1)}(\rho) \cdot F_{-1}(\rho)} \end{aligned} \quad (61)$$

Then for  $k \in \{1, \dots, n-1\}$  inductively define

$$\begin{aligned} \delta t_{k+1} &= \delta t_k + \tau_k \\ \delta\rho_{k+1} &= \delta\rho_k + \tau_k \cdot F_{\sigma(\{0, \dots, k-1\})}(\rho) \\ \sigma(k+1) &= \operatorname{argmin} \left\{ -\frac{Dh_j(\rho) \cdot \delta\rho_{k+1}}{Dh_j(\rho) \cdot F_{\sigma(\{0, \dots, k\})}(\rho)} : j \in \{1, \dots, n\} \setminus \sigma(\{1, \dots, k\}) \right\} \\ \tau_{k+1} &= -\frac{Dh_{\sigma(k+1)}(\rho) \cdot \delta\rho_{k+1}}{Dh_{\sigma(k+1)}(\rho) \cdot F_{\sigma(\{0, \dots, k\})}(\rho)} \end{aligned} \quad (62)$$

Finally, set  $\delta\rho^+ = \delta\rho_n - (\delta t_n + \tau_n) \cdot F_{+1}(\rho)$ . By construction,  $\delta\rho^- \in \Sigma'_\sigma$  and  $\delta\rho^+ = B(\delta\rho^-)$ . This strategy is succinctly summarized in pseudo-code and source code in Fig. 2; its time complexity

<sup>16</sup>This equation only holds when  $\|\delta\rho^-\|$  is small enough to ensure  $\rho^- + \delta\rho^- \in \bar{D}_{-1}$  and  $\rho^+ + \delta\rho^+ \in \bar{D}_{+1}$ ; since the B-derivative is positively homogeneous, we impose this restriction without loss of generality.

is  $O(n^2d)$  since there are  $n$  steps in the induction and each step requires  $O(n)$  dot products between  $d$ -vectors. The space complexity is  $O(d)$  since each step in the induction requires  $O(d)$  storage and data from preceding steps can be forgotten or overwritten.

We conclude by noting that, if a general-purpose algorithm is employed to solve the point location problem in  $O(dn \log n)$  time to obtain the sequence  $\sigma \in S_n$ , then the induction described in the preceding paragraph can be simplified by skipping the steps that determine  $\sigma(1)$  and  $\sigma(k+1)$  from Eqs. (61) and (62). This simplification reduces the time complexity of the induction to  $O(nd)$ , so the overall algorithm retains the  $O(dn \log n)$  time complexity of the general-purpose point-location algorithm (at the expense of the superexponential  $O(n!^d)$  space complexity of the point location algorithm). We are pessimistic these asymptotic complexities can be improved in general.

## 7 Conclusion

We constructed a representation for the *Bouligand* (or *B*-) derivative of the *piecewise- $C^r$*  (*PC<sup>r</sup>*) flow generated by an *event-selected  $C^r$*  (*EC<sup>r</sup>*) vector field and applied the representation to derive a polynomial-time algorithm to evaluate the B-derivative on a given tangent vector. Our results provide a foundation that may support future work generalizing classical analysis and synthesis techniques for smooth control systems to the class of non-smooth systems considered here. In particular, we envision applying our results to design and control the class of mechanical systems subject to unilateral constraints that arise in models of robot locomotion and manipulation.

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